

SEPARATION DICHOTOMY AND WAVEFRONTS FOR A NONLINEAR CONVOLUTION EQUATION

CARLOS GOMEZ¹, HUMBERTO PRADO² AND SERGEI TROFIMCHUK¹

¹ Instituto de Matemática y Física, Universidad de Talca
Casilla 747, Talca, Chile

² Departamento de Matemática, Universidad de Santiago de Chile
Casilla 307, Correo-2, Santiago-Chile

ABSTRACT. This paper is concerned with a scalar nonlinear convolution equation which appears naturally in the theory of traveling waves for monostable evolution models. First, we prove that each bounded positive solution of the convolution equation should either be asymptotically separated from zero or it should converge (exponentially) to zero. This dichotomy principle is then used to establish a general theorem guaranteeing the uniform persistence and existence of semi-wavefront solutions to the convolution equation. Finally, we apply our abstract results to several well-studied classes of evolution equations with asymmetric non-local and non-monotone response. We show that, contrary to the symmetric case, these equations can possess at the same time the stationary, the expansion and the extinction waves.

1. Introduction and main results. In this paper, we continue to study the nonlinear scalar convolution equation

$$\phi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g(\phi(t-s), \tau) ds, \quad t \in \mathbb{R}, \quad (1)$$

introduced in [1]. Here (X, μ) is a finite measure space, an appropriate kernel $K(s, \tau) \geq 0$ is integrable on $\mathbb{R} \times X$ with $\int_{\mathbb{R}} K(s, \tau) ds > 0$, $\tau \in X$, while measurable $g : \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$, $g(0, \tau) \equiv 0$, is continuous in ϕ for every fixed $\tau \in X$ and there exists $g'(0, \tau) > 0$. Our goal here is to establish a satisfactory criterion for the existence of semi-wavefronts (i.e. positive, bounded, and vanishing at either $+\infty$ or $-\infty$ solutions) to (1). Then in Section 5 we will apply this criterion to two non-local and asymmetric monostable evolution equations. In this way, we develop further some ideas from [19]. It should be noted that equation (1) is one of valid general forms for the description of traveling wave profiles. Other similar yet non-equivalent functional equations can be found in [2, 5, 6, 17, 21, 22].

It was shown in [1] that the characteristic function

$$\chi(z) := 1 - \int_X \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) d\mu(\tau) e^{-sz} ds.$$

plays a key role in the investigation of equation (1). In particular, the following holds (see [1, Theorem 2]):

2000 *Mathematics Subject Classification.* Primary: 34K12, 35K57; Secondary: 92D25.
Key words and phrases. Convolution, monostable equation, asymmetric non-local response.

Proposition 1. *Assume $\chi(0) < 0$. Let $\phi : \mathbb{R} \rightarrow [0, +\infty)$ be a bounded solution to equation (1). If $\phi(-\infty) = 0$ and $\phi(t) \not\equiv 0$, $t \leq t'$ for each fixed t' , then $\chi(z)$ is well defined and has a zero on some non-degenerate interval $(0, \gamma]$.*

And as we will prove below under the additional mild conditions

(C): For each $\delta > 0$ there is a measurable $C_\delta(\tau) \geq 0$ such that

$$g(u, \tau) \leq C_\delta(\tau)u, \quad u \in [0, \delta], \quad \int_X C_\delta(\tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) ds < +\infty;$$

(P): Bounded solution $\phi(t) \geq 0$ of (1) vanishes at some point only if $\phi(t) \equiv 0$, the conclusion of Proposition 1 remains true even if we replace assumption $\phi(-\infty) = 0$ by a weaker $\liminf_{t \rightarrow -\infty} \phi(t) = 0$. Moreover, in Theorem 1.2 below we prove the equivalence of these two properties for solutions of equation (1). In view of Theorem 1 and Lemma 3 from [1], this result has the following nice consequence: under a few natural restrictions on K, g , each bounded positive solution ϕ with $\liminf_{t \rightarrow -\infty} \phi(t) = 0$ converges exponentially to zero at $-\infty$.

Note that assumption **(P)** can be easily checked due to

Lemma 1.1. *Assume that there are $\tilde{X} \subset X$, $\mu(\tilde{X}) > 0$, and a measurable $A : \tilde{X} \rightarrow (0, +\infty)$ such that $\tau \in \tilde{X}$ implies (i) $g(u, \tau) = 0$ if and only if $u = 0$; (ii) $K(s, \tau) > 0$ for all $s \in (-A(\tau), A(\tau)) =: I_\tau$. Then $\phi(0) = 0$ implies $\phi(t) \equiv 0$.*

Proof. Suppose that $0 = \phi(0) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g(\phi(-s), \tau) ds$. Then we have $K(s, \tau) g(\phi(-s), \tau) = 0$ almost everywhere on $\tilde{X} \times \mathbb{R}$. Hence, for some $\tau_0 \in \tilde{X}$, we obtain that $g(\phi(-s), \tau_0) = 0$ for all $s \in I_{\tau_0}$. Thus $\phi(-s) = 0$, $s \in I_{\tau_0}$. Similarly, if $\phi(t_0) = 0$ for some $t_0 \in \mathbb{R}$, then $\phi(t) = 0$ for all t in an open neighborhood of t_0 . In consequence, the set of zeros of continuous ϕ is open and closed, and we may conclude that $\phi \equiv 0$. \square

We are ready to state our first main result:

Theorem 1.2. *Assume **(C)**, **(P)** with $\chi(0) < 0$. Then the following dichotomy holds for each bounded solution $\phi(t) \geq 0$ of (1): either $\liminf_{t \rightarrow +\infty} \phi(t) > 0$ or $\phi(+\infty) = 0$. The similar alternative is also valid at $-\infty$.*

An easy combination of results from Proposition 1 and Theorem 1.2 leads to

Corollary 1. *If $\chi(z)$ does not have any positive [negative] zero and ϕ is a positive bounded solution of (1), then $\liminf_{t \rightarrow -\infty} \phi(t) > 0$ [respectively, $\liminf_{t \rightarrow +\infty} \phi(t) > 0$]. As a consequence, equation (1) can not have positive pulse solutions (i.e. solutions satisfying $\phi(-\infty) = \phi(+\infty) = 0$).*

Proof. Since $\chi(0) < 0$ and χ is concave on its maximal domain of definition, all real zeros of χ should be of the same sign (if they exist). \square

Let ω denote either $+\infty$ or $-\infty$. By Corollary 1, we have the following point-wise persistence property: for each bounded positive solution $\phi(t)$ of Eq. (1) satisfying $\phi(-\omega) = 0$ there is some $\delta(\phi) > 0$ such that $\liminf_{t \rightarrow \omega} \phi(t) \geq \delta(\phi)$. This fact allowed us to exclude the latter inequality from the definition of semi-wavefronts (cf. with boundary conditions (1.6) in [3]). Now, in order to prove the uniform persistence (this means that the above mentioned $\delta(\phi)$ can be chosen independent of ϕ) as well as the existence of solutions to equation (1), we will impose additional conditions on its nonlinearity:

(N): N1. There exists $\tau_0 \in X, \mu(\tau_0) = 1$, such that $g(v, \tau)$ increases in v for each fixed $\tau \neq \tau_0$ and $g(v, \tau_0) > 0, v > 0$. Consider the monotone function

$$\tilde{g}(v) := \int_{X \setminus \{\tau_0\}} g(v, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) ds.$$

N2. There exists $\zeta_2 > 0$ such that $\Theta(v) := v - \tilde{g}(v)$ is strictly increasing on $[0, \zeta_2]$, and $\Theta(\zeta_2) > C \max_{v \geq 0} g(v, \tau_0)$ where $C := \int_{\mathbb{R}} K(s, \tau_0) ds$.

Set $G(v) := \Theta^{-1}(Cg(v, \tau_0))$. It is clear that $G(0) = 0, 0 < G(v) < \zeta_2, v > 0$, and that the graphs of $G(v)$ and $g(v, \tau_0)$ have similar geometrical shapes. In particular, they share the same critical points.

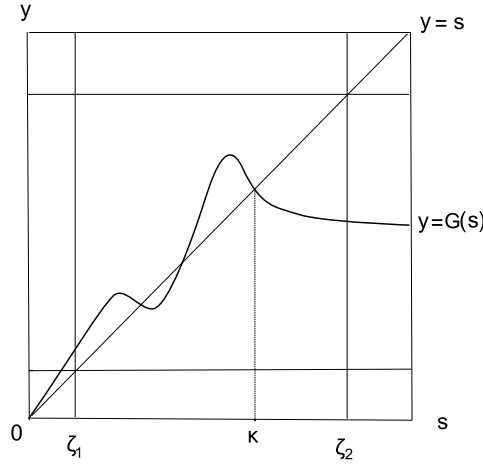


FIGURE 1. Nonlinearity G under hypotheses (N) and $\chi(0) < 0$.

If $\varphi(t) = c$ is a constant solution of (1), then $c = G(c)$ because of the relation

$$c = \tilde{g}(c) + g(c, \tau_0) \int_{\mathbb{R}} K(s, \tau_0) ds = \tilde{g}(c) + Cg(c, \tau_0) = c - \Theta(c) + Cg(c, \tau_0).$$

Several additional important properties of G are listed below:

Lemma 1.3. *Let $\chi(0) < 0$ and (C), (N) hold. Then, for some $\zeta_1 \in (0, \zeta_2)$,*

1. $G \in C(\mathbb{R}_+, \mathbb{R}_+)$ is positive for $s > 0$ and there exists $G'(0+) > 1$;
2. $G'([\zeta_1, \zeta_2]) \subseteq [\zeta_1, \zeta_2]$ and $G(\mathbb{R}_+) \subseteq [0, \zeta_2]$;
3. $\min_{s \in [\zeta_1, \zeta_2]} G(s) = G(\zeta_1)$ while $G(s) > s$ for $s \in (0, \zeta_1]$.

Proof. Let us show, for instance, that $G'(0+) > 1$. In view of (C), this derivative exists and is equal to $Cg'(0, \tau_0)/(1 - \tilde{g}'(0))$. Thus $G'(0) > 1$ if and only if $\chi(0) < 0$. Observe that $\tilde{g}'(0+) \leq 1$ since $\Theta'(0+) \geq 0$ and we do not exclude the case $G'(0+) = +\infty$. \square

Using the above framework, we can improve conclusions of Theorem 1.2:

Theorem 1.4. *Assume (N) along with all conditions of Theorem 1.2 and take $\zeta_1 > 0$ as in Lemma 1.3. Let ϕ be a positive bounded solution of equation (1). If $m = \inf_{s \in \mathbb{R}} \phi(s) < \zeta_1$ then $\lim_{t \rightarrow \omega} \phi(t) = 0$ and $\liminf_{t \rightarrow -\omega} \phi(t) > \zeta_1$ for some $\omega \in \{-\infty, +\infty\}$.*

Our third result can be considered as a further development of Theorem 6.1 from [5] which was proved for a single-point space X and under more restrictive conditions on the nonlinearity g :

Theorem 1.5. *Assume (N), that $G'(0)$ is finite and that $g(s, \tau) \leq g'(0, \tau)s$ for all $s \geq 0, \tau \in X$. If $\chi(z), \chi(0) < 0$, is defined and changes its sign on some open interval $(0, \bar{\omega})$ [respectively, on $(-\bar{\omega}, 0)$], then equation (1) has at least one semi-wavefront, with $\varphi(-\infty) = 0, \sup_{s \in \mathbb{R}} \varphi(s) \leq \zeta_2$ and $\liminf_{t \rightarrow +\infty} \varphi(t) > \zeta_1$ [respectively, with $\varphi(+\infty) = 0, \liminf_{t \rightarrow -\infty} \varphi(t) > \zeta_1$]. Moreover, if equation $G(s) = s$ has exactly two solutions 0 and κ on \mathbb{R}_+ , and the point κ is globally attracting with respect to the map $G : (0, \zeta_2] \rightarrow (0, \zeta_2]$ then $\phi(+\infty) = \kappa$.*

Remark 1. It is worth noting that the existence of $g'(0, \tau)$ (and consequently of $G'(0)$) is not at all obligatory for the existence of semi-wavefronts. Indeed, suppose that there is a measurable $l(\tau)$ satisfying $g(s, \tau) \leq l(\tau)s, s \geq 0$, and consider associated characteristics

$$\chi_l(z) := 1 - \int_X \int_{\mathbb{R}} K(s, \tau) l(\tau) d\mu(\tau) e^{-sz} ds, \quad \tilde{g}_l' := \int_{X \setminus \{\tau_0\}} \int_{\mathbb{R}} K(s, \tau) l(\tau) d\mu(\tau) ds.$$

We assume also that (N) holds, G possesses the second and the third properties of Lemma 1.3, and $\tilde{g}_l' < 1$ (this generalizes assumption $G'(0) \in \mathbb{R}$). Then all conclusions of Theorem 1.5 remain valid if we replace in its formulation χ with χ_l . See the second part of Section 4 for more details.

The paper is organized as follows. In Section 2, we prove the dichotomy principle. The first part of Section 3 shows how to avoid possible troubles with unbounded solutions of the convolution equation. The second part of the same section presents a short proof of the uniform persistence property. These preliminary results are essential for proving the existence theorem in Section 4. Finally, several applications are considered in the last section of the paper. Associated characteristic equation is analyzed in Appendix.

2. The proof of the dichotomy principle (Theorem 1.2).

1. Let $\phi(t)$ be a bounded solution of (1). It is easy to see that $\phi(t)$ is uniformly continuous on \mathbb{R} . Indeed, setting $\delta = |\phi|_\infty$, we find that

$$\begin{aligned} |\phi(t+h) - \phi(t)| &\leq \int_X d\mu(\tau) \int_{\mathbb{R}} |K(s+h, \tau) - K(s, \tau)| g(\phi(t-s), \tau) ds \\ &\leq |\phi|_\infty \int_X C_\delta(\tau) d\mu(\tau) \int_{\mathbb{R}} |K(s+h, \tau) - K(s, \tau)| ds =: |\phi|_\infty \sigma_\delta(h), \end{aligned}$$

where $\lim_{h \rightarrow 0} \sigma_\delta(h) = 0$ because of the continuity of translation in $L_1(\mathbb{R})$ and the Lebesgue's dominated convergence theorem.

2. Next we prove an analog of Proposition 1 when $\phi(+\infty) = 0$ and ϕ is bounded and positive. We have

$$\phi(-t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g(\phi(-t-s), \tau) ds, \quad t \in \mathbb{R}.$$

Set $\psi(t) := \phi(-t)$, then $\psi(-\infty) = 0$ and

$$\psi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(-s, \tau) g(\psi(t-s), \tau) ds. \quad (2)$$

Let $\chi(z)$ [$\chi_1(z)$] be characteristic equation for Eq. (1) [Eq. (2), respectively]. We have

$$\begin{aligned}\chi_1(z) &= 1 - \int_X \int_{\mathbb{R}} K(-s, \tau) g'(0, \tau) d\mu(\tau) e^{-sz} ds \\ &= 1 - \int_{\mathbb{R}} \int_X K(s, \tau) g'(0, \tau) d\mu(\tau) e^{sz} ds = \chi(-z)\end{aligned}$$

and thus $\chi_1(0) = \chi(0) < 0$. By Proposition 1, $\chi_1(z)$ has at least one positive root. Therefore $\chi(z)$ has at least one negative zero.

3. Now, let suppose that $\limsup_{t \rightarrow +\infty} \phi(t) = S > 0$ and $\liminf_{t \rightarrow +\infty} \phi(t) = 0$. Since $\chi(0) < 0$ and χ is concave on its maximal domain of definition, all real zeros of χ should be of the same sign (if they exist). Suppose that χ does not have any real negative [respectively, positive] root. For a fixed $j > S^{-1}$ there exists a sequence of intervals $[p_i, q_i]$, $\lim p_i = +\infty$, such that $\phi(p_i) = 1/j$, $\lim \phi(q_i) = 0$ [respectively, $\phi(q_i) = 1/j$, $\lim \phi(p_i) = 0$] and $\phi(t) \leq 1/j$, $t \in [p_i, q_i]$. Note that $\limsup_{i \rightarrow +\infty} (q_i - p_i) = +\infty$. Indeed, otherwise we can suppose that $\lim_{i \rightarrow +\infty} (q_i - p_i) = \sigma > 0$. By the pre-compactness of $\{\phi(t + s); s \in \mathbb{R}\}$ in the compact-open topology of $C(\mathbb{R})$, the sequence $w_i(t) := \phi(t + p_i)$ [respectively, $w_i(t) := \phi(t + q_i)$] of solutions to Eq. (1) contains a subsequence converging to a non-negative bounded function $w_*(t)$ such that $w_*(0) = 1/j$, $w_*(\sigma)w_*(-\sigma) = 0$. Since, due to the Lebesgue's dominated convergence theorem, $w_*(t)$ satisfies (1) as well, this contradicts to (P). Thus $q_i - p_i \rightarrow +\infty$ and we can suppose that $w_i(t)$ has a subsequence converging to a bounded positive solution $w_*(t)$ of (1) satisfying $0 < w_*(t) \leq 1/j$ for all $t \geq 0$ [respectively, for all $t \leq 0$]. Since $w_*(+\infty) = 0$ [respectively, $w_*(-\infty) = 0$] is impossible due to Proposition 1 and the second step of the proof, we conclude that $0 < S^* = \limsup_{t \rightarrow +\infty} w_*(t) \leq 1/j$ [respectively, $0 < S^* = \limsup_{t \rightarrow -\infty} w_*(t) \leq 1/j$]. Let $r_i \rightarrow +\infty$ [respectively, $r_i \rightarrow -\infty$] be such that $w_*(r_i) \rightarrow S^*$, then $w_*(t + r_i)$ has a subsequence converging to a positive solution $\zeta_j : \mathbb{R} \rightarrow [0, 1/j]$ of (1) such that $\max_{t \in \mathbb{R}} \zeta_j(t) = \zeta_j(0) = S^* \leq 1/j$. Now, let us consider $y_j(t) = \zeta_j(t)/\zeta_j(0)$. Each y_j satisfies

$$y_j(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) a_j(t - s, \tau) y_j(t - s) ds, \quad (3)$$

where $a_j(t, \tau) = g(\zeta_j(t), \tau)/\zeta_j(t)$. We claim that $\{y_j(t)\}$ has a subsequence converging to a continuous solution $y_* : \mathbb{R} \rightarrow [0, 1]$, $y_*(0) = 1$, of equation

$$y_*(t) = \int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) y_*(t - s) ds. \quad (4)$$

Indeed, the sequence $\{y_j(t)\}_{j=1}^{+\infty}$ is equicontinuous because of

$$\begin{aligned}|y_j(t + h) - y_j(t)| &\leq \int_X d\mu(\tau) \int_{\mathbb{R}} a_j(t - s) y_j(t - s) |K(s + h, \tau) - K(s, \tau)| ds \\ &\leq \int_X d\mu(\tau) \int_{\mathbb{R}} a_j(t - s) |K(s + h, \tau) - K(s, \tau)| ds \leq \sigma_1(h),\end{aligned}$$

where σ_δ was defined on step 1. In addition,

$$\left| \int_{\mathbb{R}} K(s, \tau) a_j(t - s, \tau) y_j(t - s) ds \right| \leq C_1(\tau) \int_{\mathbb{R}} K(s, \tau) ds \in L_1(X),$$

so that, by the Lebesgue's dominated convergence theorem, we can pass to the limit (as $j \rightarrow \infty$) in (3). Hence, our claim is proved.

4. The proof of Theorem 1.2 will be finalized, if we show that (4) cannot have any nontrivial continuous solution $y_* \geq 0$. Since

$$\int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) ds > 1$$

there exists $N > 0$ such that

$$\rho := \int_X g'(0, \tau) d\mu(\tau) \int_{-N}^N K(s, \tau) ds > 1.$$

Integrating equation (4) between t' and $t > t'$, we obtain

$$\begin{aligned} \int_{t'}^t y_*(v) dv &\geq \int_X g'(0, \tau) d\mu(\tau) \int_{-N}^N K(s, \tau) \int_{t'}^t y_*(v-s) dv ds \\ &= \int_X g'(0, \tau) d\mu(\tau) \int_{-N}^N K(s, \tau) \left(\int_{t'-s}^{t'} + \int_{t'}^t + \int_t^{t-s} \right) y_*(v) dv ds, \end{aligned}$$

from which

$$\int_{t'}^t y_*(v) dv \leq \frac{2 \int_X \int_{-N}^N |s| K(s, \tau) g'(0, \tau) ds d\mu(\tau)}{\int_X \int_{-N}^N K(s, \tau) g'(0, \tau) ds d\mu(\tau) - 1}, \quad t' < t.$$

Therefore $y_* \in L_1(\mathbb{R})$. Now we easily get a contradiction by integrating (4) over the real line:

$$0 < \int_{\mathbb{R}} y_*(v) dv = \left[\int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) ds \right] \int_{\mathbb{R}} y_*(v) dv.$$

Hence, the dichotomy principle of Theorem 1.2 is established at $+\infty$. The other case can be reduced to the previous one by doing the change of variables $\psi(t) := \phi(-t)$ and considering equation (2) with χ_1 instead of (1) with χ . \square

3. The uniform permanence property.

3.1. The uniform boundedness of solutions. It should be noted that, in general, equation (1) might have unbounded continuous solutions. Corresponding examples can be constructed by taking appropriate linear $g(u, \tau)$. Nevertheless, as we show in the continuation, with conditions (N) and $\chi(0) < 0$ being assumed, it is easy to avoid eventual troubles with unbounded solutions in the following two ways:

Modification of the convolution equation. Consider

$$\bar{g}(u, \tau) = \min\{g(u, \tau), g(\zeta_2, \tau)\}, \quad \tau \neq \tau_0, \quad \bar{g}(u, \tau_0) := g(u, \tau_0)$$

and

$$\bar{g}(v) = \int_{X \setminus \{\tau_0\}} \bar{g}(v, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) ds.$$

Then $\bar{\Theta}(s) := s - \bar{g}(s)$ is a strictly increasing function. Indeed, $\bar{\Theta}(s) = \Theta(s)$, $0 \leq s \leq \zeta_2$, and we know that $\Theta(s)$ strictly increases in $[0, \zeta_2]$. Furthermore, for $s \geq \zeta_2$, we have $\bar{\Theta}(s) = s - \bar{g}(s) = s - \bar{g}(\zeta_2)$ where $\bar{g}(\zeta_2)$ is a constant. Hence $\bar{\Theta}(s)$ is strictly increasing on \mathbb{R}_+ . If we set $\bar{G}(v) = \bar{\Theta}^{-1}(C\bar{g}(v, \tau_0))$, we find that $\bar{G}(v) = G(v) \leq \zeta_2$ for $v \geq 0$.

Let us consider now a modified convolution equation

$$\phi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) \bar{g}(\phi(t-s), \tau) ds.$$

Each its solution $\phi(t)$ is bounded;

$$\phi(t) \leq \bar{g}(\zeta_2) + C \max_{v \geq 0} g(v, \tau_0) < \bar{g}(\zeta_2) + \Theta(\zeta_2) = \zeta_2.$$

The latter estimate assures that $\phi(t)$ simultaneously satisfies (1).

Subexponential solutions. Assume additionally that

$$g(u, \tau) \leq g'(0, \tau)u, \quad u \geq 0, \quad \text{for each } \tau \neq \tau_0. \quad (5)$$

If, for some $\lambda > 0$, ϕ satisfies (1) and $\phi(t) \leq \delta e^{\lambda t}$, $t \in \mathbb{R}$, then

$$\phi(t) \leq \int_{X \setminus \{\tau_0\}} d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) \phi(t-s) ds + \rho \quad (6)$$

where $\rho := \sup_{u \geq 0} g(u, \tau_0) \int_{\mathbb{R}} K(s, \tau_0) ds \leq \Theta(\zeta_2)$. Suppose, in addition, that

$$\theta := \int_{X \setminus \{\tau_0\}} d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) e^{-\lambda s} ds < 1$$

and $\gamma := \int_{X \setminus \{\tau_0\}} d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) ds < 1$. The first inequality holds automatically if $\chi(\lambda) = 0$ because of $\int_{\mathbb{R}} K(s, \tau_0) g'(0, \tau_0) e^{-\lambda s} ds > 0$. Similarly, since $\gamma = \tilde{g}'(0)$, the second inequality holds whenever $G'(0+)$ is finite.

Lemma 3.1. *If (5) holds, $\chi(\lambda) = 0$ and $G'(0)$ is a finite number then each solution $\phi(t) \leq \delta e^{\lambda t}$ of (1) is bounded. In fact,*

$$0 \leq \phi(t) \leq \min\{\zeta_2, \sup_{u \geq 0} g(u, \tau_0) \frac{G'(0)}{g'(0, \tau_0)}\}, \quad t \in \mathbb{R}.$$

Proof. Using $\phi(t) \leq \delta e^{\lambda t}$ in (6) and arguing by induction, we find that

$$\phi(t) \leq \delta e^{\lambda t} \theta^n + \rho + \rho\gamma + \rho\gamma^2 + \dots + \rho\gamma^n.$$

Then, by passing to the limit as $n \rightarrow \infty$, we obtain the required estimate. We recall here that $\gamma = \tilde{g}'(0)$, $G'(0) = C g'(0, \tau_0) / (1 - \tilde{g}'(0))$ and $C = \int_{\mathbb{R}} K(s, \tau_0) ds$. The inequality $\phi(t) \leq \zeta_2$ follows from Lemma 3.2 proved in continuation. \square

3.2. The proof of the uniform persistence (Theorem 1.4). Let ϕ a bounded positive solution of the equation (1). Set

$$0 \leq m := \inf_{t \in \mathbb{R}} \phi(t) \leq \sup_{t \in \mathbb{R}} \phi(t) =: M < +\infty.$$

Lemma 3.2. $[m, M] \subseteq G([m, M])$.

Proof. Let $\{t_j\}$ be such that $M_j := \phi(t_j) \rightarrow M$. We have

$$\begin{aligned} \phi(t_j) &= M_j \leq \int_X \max_{v \in [m, M]} g(v, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) ds \\ &= \max_{v \in [m, M]} \int_{X \setminus \{\tau_0\}} g(v, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) ds + \max_{v \in [m, M]} g(v, \tau_0) \int_{\mathbb{R}} K(s, \tau_0) ds \\ &= \tilde{g}(M) + \max_{v \in [m, M]} g(v, \tau_0) \int_{\mathbb{R}} K(s, \tau_0) ds. \end{aligned}$$

Thus $M \leq \max_{v \in [m, M]} G(v)$. Similarly, $m \geq \min_{v \in [m, M]} G(v)$. \square

Now, assumption (\mathbf{N}) , $G'(0) > 1$ and $m < \zeta_1$ yield $m = 0$, cf. Fig. 1. Hence, due to the positivity of $\phi(t)$, there exists $\omega \in \{-\infty, +\infty\}$ such that $\liminf_{t \rightarrow \omega} \phi(t) = 0$. Then, applying Theorem 1.2 and Corollary 1, we find that $\phi(\omega) = 0$ and $\mu := \liminf_{t \rightarrow -\omega} \phi(t) > 0$. Making use of our standard limiting solution argument, we see that, for some $t_j \rightarrow -\omega$, the sequence $\phi(t + t_j)$ is converging in the compact-open topology of $C(\mathbb{R})$ to some function $\phi_1(t)$, $\mu := \inf_{t \in \mathbb{R}} \phi_1(t) \leq \sup_{t \in \mathbb{R}} \phi_1(t) \leq M$ solving equation (1). By Lemma 3.2, we have $[\mu, M] \subseteq G([\mu, M])$ which implies $\mu > \zeta_1$. \square

Remark 2. The last argument in the proof of Lemma 3.2 shows also that $[m', M'] \subseteq G([m', M'])$, where $m' := \liminf_{t \rightarrow \omega} \phi(t) \leq \limsup_{t \rightarrow \omega} \phi(t) =: M'$ and $\omega \in \{-\infty, +\infty\}$.

4. The proof of the existence. Throughout all this section, we are assuming that (\mathbf{N}) holds, $\chi(0) < 0$ and

$$g(s, \tau) \leq g'(0, \tau)s \text{ for all } s \geq 0, \tau \in X. \quad (7)$$

1. For a moment, let us suppose additionally that

(\mathbf{L}) $g : (0, \infty) \times X \rightarrow (0, +\infty)$ is bounded and uniformly linear in some right neighborhood of the origin: $g(s, \tau) = g'(0, \tau)s$, $s \in [0, \delta)$, $\tau \in X$.

Let $\lambda \in (0, \bar{\omega})$ be the leftmost positive solution of equation $\chi(z) = 0$, and set

$$\begin{aligned} X &:= \{\varphi \in C(\mathbb{R}, \mathbb{R}) : \|\varphi\| = \sup_{s \leq 0} e^{-0.5\lambda s} |\varphi(s)| + \sup_{s \geq 0} e^{-\nu s} |\varphi(s)| < +\infty\}; \\ \mathfrak{K} &:= \{\varphi \in X; \phi^-(t) = \delta e^{\lambda t} (1 - e^{\epsilon t}) \chi_{\mathbb{R}_-}(t) \leq \varphi(t) \leq \delta e^{\lambda t} = \phi^+(t), t \in \mathbb{R}\}, \end{aligned}$$

where $\epsilon > 0$ and $\nu := \lambda + \epsilon < \bar{\omega}$ are such that $\chi(\nu) > 0$. We want to prove the existence of fixed points φ , $\varphi \in \mathfrak{K}$, $\sup_{s \in \mathbb{R}} \varphi(s) < +\infty$, to the operator

$$\mathcal{A}\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g(\varphi(t-s), \tau) ds.$$

A formal linearization of \mathcal{A} along the trivial steady state is given by

$$L\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) \varphi(t-s) ds.$$

$$\begin{aligned} \text{We have that } L\phi^+(t) &= \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) \delta e^{\lambda(t-s)} ds \\ &= \delta e^{\lambda t} \int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) e^{-\lambda s} ds = \delta e^{\lambda t} = \phi^+(t). \end{aligned}$$

On the other hand, $L\phi^-(t) > \phi^-(t)$, $t \in \mathbb{R}$. Indeed, we have, for a fixed $t \leq 0$,

$$\begin{aligned} \delta^{-1} L\phi^-(t) &= \int_X d\mu(\tau) \int_t^{+\infty} K(s, \tau) g'(0, \tau) (e^{\lambda(t-s)} - e^{\nu(t-s)}) ds \\ &\geq \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) (e^{\lambda(t-s)} - e^{\nu(t-s)}) ds \\ &= e^{\lambda t} - e^{\nu t} (1 - \chi(\nu)) = e^{\lambda t} - e^{\nu t} + e^{\nu t} \chi(\nu) > e^{\lambda t} - e^{\nu t} = \delta^{-1} \phi^-(t). \end{aligned}$$

Lemma 4.1. \mathfrak{K} is a closed, bounded, convex subset of X and $\mathcal{A} : \mathfrak{K} \rightarrow \mathfrak{K}$ is a completely continuous map.

Proof. It is clear that \mathfrak{K} is a closed, bounded, convex subset of X . To prove that $\mathcal{A}(\mathfrak{K}) \subseteq \mathfrak{K}$, we observe first that, for $\varphi \in \mathfrak{K}$,

$$\mathcal{A}\varphi(t) \leq \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) \varphi(t-s) ds = L\varphi(t) \leq L\varphi^+(t) = \varphi^+(t).$$

Next, if for some u we have that $0 < \varphi^-(u) \leq \varphi(u)$, then $u < 0$ so that $\varphi(u) \leq \delta e^{\lambda u} \leq \delta$, which implies that $g(\varphi(u), \tau) = g'(0, \tau)\varphi(u)$. If $\varphi^-(u) = 0$ then $g(\varphi(u), \tau) \geq g'(0, \tau)\varphi^-(u) = 0$. In either case,

$$\mathcal{A}\varphi(t) \geq \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) g'(0, \tau) \varphi^-(t-s) ds = L\varphi^-(t) > \varphi^-(t).$$

Now, we claim that $\mathcal{A}\mathfrak{K}$ is a precompact subset of \mathfrak{K} . Indeed, the convergence in \mathfrak{K} is the uniform convergence on compact subsets of \mathbb{R} . On the other hand, the set of functions from $\mathcal{A}\mathfrak{K}$ restricted on every fixed compact interval $[-k, k]$ is obviously uniformly bounded and is also equicontinuous in virtue of the estimation (uniform with respect to $t \in [-k, k], \phi \in \mathfrak{K}$): $|\mathcal{A}\phi(t+h) - \mathcal{A}\phi(t)| \leq$

$$\delta e^{\lambda k} \int_X g'(0, \tau) d\mu(\tau) \int_{\mathbb{R}} |K(s+h, \tau) - K(s, \tau)| e^{-\lambda s} ds \rightarrow 0, \quad h \rightarrow 0. \quad (8)$$

Finally, the continuity of \mathcal{A} in \mathfrak{K} can be easily established by using the dominated convergence theorem and the compactness property of \mathcal{A} . \square

Then Lemmas 3.1, 3.2, 4.1 and the Schauder's fixed point theorem yields

Theorem 4.2. *Assume (L) and let λ be the leftmost positive zero of χ . Then \mathcal{A} has at least one fixed point ϕ in \mathfrak{K} . If $G'(0)$ is a finite number then $|\phi|_\infty := \sup_{s \in \mathbb{R}} \phi(s)$ is also finite and $|\phi|_\infty < \zeta_2$. Moreover, if the point κ is globally attracting with respect to the map $G : (0, \zeta_2] \rightarrow (0, \zeta_2]$ then $\phi(+\infty) = \kappa$.*

It should be noted that the last statement of this theorem is a straightforward consequence of Remark 2 (see also [11] where various conditions assuring the global stability property of G are given).

2. Next we show how to reduce the general situation to the case studied in the first part of this section. Consider the sequence of measurable functions

$$\gamma_n(s, \tau) := \begin{cases} g'(0, \tau)s, & \text{for } s \in [0, 1/n], \\ \max\{g'(0, \tau)/n, g(s, \tau)\}, & \text{when } s \geq 1/n, \end{cases}$$

all of them continuous in s for each fixed τ and satisfying hypothesis (L) with $\delta = 1/n$. Note that $\gamma_n(s, \tau)$ converges uniformly to $g(s, \tau)$ on \mathbb{R}_+ for every fixed τ . Next, set $X' := X \setminus \{\tau_0\}$ and consider continuous increasing functions

$$\tilde{g}_n(v) := \int_{X'} \gamma_n(v, \tau) d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) ds, \quad n = 1, 2, 3 \dots$$

Since $\gamma_{n+1}(s, \tau) \leq \gamma_n(s, \tau)$, $n = 1, 2, 3 \dots$, the sequence $\{\tilde{g}_n\}$ is monotone. Now, for each fixed $v \geq 0$, we have that $\lim_{n \rightarrow +\infty} \tilde{g}_n(v) = \tilde{g}(v)$ where \tilde{g} was defined in N2. Observe that \tilde{g} is also continuous and therefore, by Dini's monotone convergence theorem, \tilde{g}_n converges to \tilde{g} uniformly on compacts.

Lemma 4.3. *Let $G'(0) > 1$ be a finite number. Then $\Theta_n(v) := v - \tilde{g}_n(v)$ is strictly increasing in v . Furthermore, $G_n(v) := \Theta_n^{-1}(C\gamma_n(v, \tau_0))$ converges to $G(v)$ uniformly on $[0, \zeta_2]$ and $G'_n(0) = G'(0) > 1$. Finally, equation $G_n(c) = c$ does not have solutions on $(0, \zeta_1]$.*

Proof. Set $w(\tau) := \int_{\mathbb{R}} K(s, \tau) ds$. Since $G'(0)$ is finite, we have that

$$\tilde{g}'(0) = \int_{X'} g'(0, \tau) w(\tau) d\mu(\tau) < 1.$$

Now, if $v \in [0, 1/n]$ then $\tilde{g}_n(v) = \tilde{g}'(0)v$ and therefore $\tilde{g}_n(v_2) - \tilde{g}_n(v_1) = \tilde{g}'(0)(v_2 - v_1) < v_2 - v_1$ for $0 \leq v_1 < v_2 \leq 1/n$.

Next, for $1/n \leq v_1 < v_2$ we consider the following measurable subsets of X' :

$$A_j := \left\{ \tau \in X' : g(v_j, \tau) \leq \frac{g'(0, \tau)}{n} \right\}, \quad B_j := \left\{ \tau \in X' : g(v_j, \tau) > \frac{g'(0, \tau)}{n} \right\}.$$

Clearly, $B_j = X' \setminus A_j$, $A_2 \subset A_1$, $B_1 \subset B_2$ and $B_2 \setminus B_1 = A_1 \setminus A_2$. We have

$$\begin{aligned} \tilde{g}_n(v_2) - \tilde{g}_n(v_1) &= \int_{B_2 \setminus B_1} (g(v_2, \tau) - \frac{g'(0, \tau)}{n}) w(\tau) d\mu(\tau) + \\ &\int_{B_1} (g(v_2, \tau) - g(v_1, \tau)) w(\tau) d\mu(\tau) \leq \int_{B_2} (g(v_2, \tau) - g(v_1, \tau)) w(\tau) d\mu(\tau) \leq \\ &\int_{X'} (g(v_2, \tau) - g(v_1, \tau)) w(\tau) d\mu(\tau) = \tilde{g}(v_2) - \tilde{g}(v_1) < v_2 - v_1. \end{aligned}$$

Finally, consider $v_1 < 1/n < v_2$. Then

$$\tilde{g}_n(v_2) - \tilde{g}_n(v_1) = \tilde{g}_n(v_2) - \tilde{g}_n(1/n) + \tilde{g}_n(1/n) - \tilde{g}_n(v_1) < v_2 - 1/n + 1/n - v_1 = v_2 - v_1.$$

This proves that Θ_n are strictly increasing. Moreover, since clearly $\Theta_n(\zeta_2) > \max_{s \geq 0} C\gamma_n(v, \tau_0)$ for all large n , the functions G_n are well defined. The second conclusion of the lemma follows now immediately from the uniform convergence properties of the sequences $\{\gamma_n(v, \tau_0)\}$, $\{\tilde{g}_n(v)\}$. Note also that $G_n(v) = G'(0)v$ in some small neighborhood U_n of $v = 0$. Finally, to prove the last conclusion of the lemma, we observe that $G_n(c) = c$ implies

$$c = \int_X \gamma_n(c, \tau) w(\tau) d\mu(\tau) \geq \int_X g(c, \tau) w(\tau) d\mu(\tau) = \tilde{g}(c) + g(c, \tau_0) w(\tau_0).$$

In this way, $\Theta(c) \geq g(c, \tau_0) w(\tau_0)$ so that $c \geq G(c)$. Since $G(s) > s$ on $[0, \zeta_1]$ (see Lemma 1.3.3), we conclude that also $G_n(s) > s$ for $s \in [0, \zeta_1]$. \square

Corollary 2. *For all sufficiently large n , and with the same ζ_1 and ζ_2 as in Lemma 1.3, each G_n possesses all three properties listed in Lemma 1.3.*

Hence, for each large n , Corollary 2, Theorems 4.2 and 1.4 guarantee the existence of a positive continuous function $\varphi_n(t)$ such that $\varphi_n(-\infty) = 0$, $\liminf_{t \rightarrow +\infty} \varphi_n(t) \geq \zeta_1$, $\varphi_n(t) \leq \zeta_2$, $t \in \mathbb{R}$, and

$$\varphi_n(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} K(s, \tau) \gamma_n(\varphi_n(t-s), \tau) ds.$$

Since the shifted functions $\varphi_n(s+a)$ satisfy the same integral equation, we can assume that $\varphi_n(0) = 0.5\zeta_1$. Furthermore, similarly to (8) we can show that the sequence $\{\varphi_n\}$ is equicontinuous on \mathbb{R} . Consequently there exists a subsequence $\{\varphi_{n_j}\}$ which converges uniformly on compacts to some bounded element $\phi \in C(\mathbb{R}, \mathbb{R})$. By the Lebesgue's dominated convergence theorem, ϕ satisfies equation (1). Finally, notice that $\phi(0) = 0.5\zeta_1$ and thus $\phi(-\infty) = 0$ and $\liminf_{t \rightarrow +\infty} \phi(t) > \zeta_1$ (by Theorem 1.4). This finalizes the proof of Theorem 4.2 when $\chi(z)$ has a positive zero. Its statement for $\chi(z)$ having a negative zero is immediate after the change of variables $\psi(t) = \phi(-t)$. \square

5. Applications.

5.1. Co-existence of expansion and extinction waves in evolution equations with asymmetric non-local response [1, 10, 13, 15, 18, 19].

Here we complement studies [1, 18, 19] concerning positive bounded wavefronts $u(x, t) = \phi(x + ct)$ for the non-local delayed reaction-diffusion equation

$$u_t(t, x) = u_{xx}(t, x) - f(u(t, x)) + \int_{\mathbb{R}} K(x - y)g(u(t - h, y))dy, \quad u \geq 0, \quad (9)$$

where

(\mathcal{F}) *locally Lipschitzian function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $f'(0) > f(0) = g(0) = 0$, is strictly increasing and $f(+\infty) > \sup_{s \geq 0} g(s)$. In addition, $f'(0) < g'(0) < +\infty$ and $g(t) > 0$, $t > 0$. Kernel $K \geq 0$ is normalized by $\int_{\mathbb{R}} K(s)ds = 1$.*

We admit spatial asymmetry of equation (9) by considering non-even kernels. Due to this circumstance, the concept of wavefront needs some clarification. Indeed, in the symmetric case, the following two equivalent definitions have been commonly used: 1) *wavefront $u(x, t) = \phi(x - ct)$ is a positive classical solution of (9) satisfying $\phi(-\infty) = \kappa$, $\phi(+\infty) = 0$, e.g. see [3, 12]; 2) *wavefront $u(x, t) = \psi(x + ct)$ is a positive classical solution satisfying $\psi(-\infty) = 0$, $\psi(+\infty) = \kappa$, e.g. see [10, 18]. If $K(s) \equiv K(-s)$, both definitions define the same object since wavefront $\phi(x - ct)$ generates wavefront $\psi(x + ct) := \phi(-(x + ct))$. Moreover, the propagation speed c should be positive in each of the above definitions if K is an even function. Therefore, from the biological point of view the both type of wavefronts can be interpreted as *the expansion fronts*: they converge to the positive equilibrium at each fixed position x as $t \rightarrow +\infty$.**

Taking into account the above discussion, we will use more general definition adapted to the possible asymmetry K : *Bounded positive classical solution $u(x, t) = \phi(x + ct)$ of equation (9) is a semi-wavefront if either $\phi(-\infty) = 0$ or $\phi(+\infty) = 0$. The prefix *semi* means here that, contrary to the wavefronts, the convergence of $\phi(t)$ at the complementary end of \mathbb{R} is not mandatory. It is clear that $u(x, t) = \phi(x + ct)$ is a semi-wavefront if and only if $\phi(t)$ is a positive bounded C^2 -solution of the integro-differential equation*

$$y''(t) - cy'(t) - f(y(t)) + \int_{\mathbb{R}} K(s)g(y(t - s - ch))ds = 0, \quad (10)$$

which vanishes either at $-\infty$ or at $+\infty$. By abusing the notation, we still call such a solution $y = \phi(t)$ a semi-wavefront. Equation (10) can be written as

$$y''(t) - cy'(t) - \beta y(t) + f_{\beta}(y(t)) + \int_{\mathbb{R}} k_h(w)g(y(t - w))dw = 0, \quad t \in \mathbb{R},$$

where $k_h(w) = K(w - ch)$ and $f_{\beta}(s) = \beta s - f(s)$ for some $\beta > 0$. Then the wave profile ϕ solves the equation

$$\phi(t) = \frac{1}{\sigma(c)} \left(\int_{-\infty}^t e^{\nu(t-s)} (\mathcal{G}\phi)(s)ds + \int_t^{+\infty} e^{\mu(t-s)} (\mathcal{G}\phi)(s)ds \right),$$

where $\sigma(c) = \sqrt{c^2 + 4\beta}$, $\nu < 0 < \mu$ are the roots of $z^2 - cz - \beta = 0$ and $(\mathcal{G}\phi)(t) := \int_{\mathbb{R}} k_h(s)g(\phi(t - s))ds + f_{\beta}(\phi(t))$, e.g. see [1]. In other words,

$$\phi(t) = (\mathcal{K} * k_h) * g(\phi)(t) + \mathcal{K} * f_{\beta}(\phi)(t),$$

where $\mathcal{K}(s) = e^{\nu s}/\sigma(c)$ for $s \geq 0$, $\mathcal{K}(s) = e^{\mu s}/\sigma(c)$ for $s \leq 0$, and consequently $\int_{\mathbb{R}} \mathcal{K}(s) ds = 1/\beta$. We may invoke now Theorems 1.4, 1.5 where $X = \{\tau_0, \tau_1\}$ and

$$K(s, \tau) = \begin{cases} (\mathcal{K} * k_h)(s), & \tau = \tau_0, \\ \mathcal{K}(s), & \tau = \tau_1, \end{cases} \quad g(s, \tau) = \begin{cases} g(s), & \tau = \tau_0, \\ f_\beta(s), & \tau = \tau_1. \end{cases}$$

Observe that the functions $g(u, \tau_j), K(s, \tau_j)$ meet (N). Indeed, there exists ζ_2 such that $f(\zeta_2) > \sup_{s \geq 0} g(s)$. Then we can take β large enough to have the function $\tilde{g}(s) = f_\beta(s)/\beta = s - f(s)/\beta$ increasing on $[0, \zeta_2]$. Next, $\Theta(v) := v - \tilde{g}(v) = f(v)/\beta$ is strictly increasing by (F) and $G(s) = f^{-1}(g(s))$ is well defined. Finally,

$$\chi(z) = \frac{-\chi_1(z, c)}{\beta + cz - z^2}, \text{ where } \chi_1(z, c) = z^2 - cz - f'(0) + g'(0)e^{-zh} \int_{\mathbb{R}} K(s)e^{-zs} ds.$$

Analyzing the mutual position of real zeros of $\chi_1(z, c)$ and their dependence on the parameter c , we establish in Appendix the existence of two real extended numbers $c_*^- < c_*^+$ called *the critical speeds* such that, for every $c \in (-\infty, c_*^-] \cup [c_*^+, +\infty)$, equation $\chi_1(\lambda, c) = 0$ either (i) has exactly two real roots $\lambda_1(c) \leq \lambda_2(c)$ or (ii) has exactly one real root $\lambda_1(c)$. Furthermore, each $\lambda_j(c)$ is positive if $c \geq c_*^+$ and is negative if $c \leq c_*^-$. If $c \in (c_*^-, c_*^+)$, then $\chi_1(z, c) > 0$ for all admissible z . The critical speed $c_*^+ [c_*^-]$ is finite if and only if $\chi_1(\lambda, c)$ is finite for some $\lambda > 0$ [respectively, with some $\lambda < 0$]. If the integral in χ_1 diverges for all $z > 0$ [for all $z < 0$], we set $c_*^+ = +\infty$ [respectively, $c_*^- = -\infty$].

Remark 3. The above definition of c_*^\pm generalizes the concept of critical speeds $c_*, c_\# \geq 0$ from [19]. In particular, $c_* = c_*^+$, $c_\# = c_*^-$ if $c_*^- \geq 0$ and $c_\# = 0$, $c_* = \max\{0, c_*^+\}$ if $c_*^- < 0$. Thus Theorem 5.1 below gives a global (i.e. including all $c \in \mathbb{R}$) perspective on the existence/persistence results in [19].

Applied to equation (10), Theorem 1.5 yields the following extension of [18, Theorem 4.2b], [15, Theorem 1.1] and [19, Theorem 4]:

Theorem 5.1. *Assume (F) and $g(s) \leq g'(0)s$, $f(s) \geq f'(0)s$ for all $s \geq 0$. Then equation (10) has at least one semi-wavefront $u = \phi_c(x + ct) \leq \zeta_2$ for each $c \in (-\infty, c_*^-] \cup [c_*^+, +\infty)$. Moreover, if $c \leq c_*^-$ then $\phi_c(+\infty) = 0$ and $\liminf_{s \rightarrow -\infty} \phi_c(s) > \zeta_1$. Similarly, if $c \geq c_*^+$ then $\phi_c(-\infty) = 0$ and $\liminf_{s \rightarrow +\infty} \phi_c(s) > \zeta_1$. Next, if equation $f(s) = g(s)$ has only two solutions: $0, \kappa$, with κ being globally attracting with respect to the map $f^{-1} \circ g : (0, \zeta_2] \rightarrow (0, \zeta_2]$, then each of these semi-wavefronts is in fact a wavefront.*

Proof. For a fixed $c' \in \mathbb{R} \setminus [c_*^-, c_*^+]$, this result follows from Theorem 1.5 since the equation $\chi_1(z, c') = 0$ has at least one real root in the interior of the domain of definition of $\chi_1(\cdot, c')$. Now, if $c' \in \{c_*^-, c_*^+\}$ is finite, we obtain a semi-wavefront $\phi_{c'}$ as a limit of profiles ϕ_{c_j} where either $c_j \uparrow c_*^-$ or $c_j \downarrow c_*^+$. See Section 4.2 above or [19, Section 6, Case II] for more details. \square

We observe that each possible mutual position of $c_*^- \leq c_*^+$ and 0 is possible. For instance, if $K(s) = e^{-(s+\rho)^2}/\sqrt{4\pi}$, $h = 2$, $g'(0) = 2 > f'(0) = 1$, then $c_*^+ = -c_*^- = 0.79$ for $\rho = 0$ (symmetric case), while $c_*^+ = 2.7$, $c_*^- = 0.7\dots$ for $\rho = 5$ (asymmetric case). In particular, if $\rho = 5$ then equation (10) has at least one *stationary* (i.e. propagating at the velocity $c = 0$) semi-wavefront. In the case when c_*^-, c_*^+ are of the same sign, an interesting (by its possible biological interpretation) phenomenon occurs: equation (10) can possess the *extinction* waves. Indeed, if $0 < c < c_*^-$ then the wave $u(x, t) = \phi(x + ct)$ converges to 0 at each position x

as $t \rightarrow +\infty$. Analogously, for each $x \in \mathbb{R}$, we have $\lim_{t \rightarrow -\infty} u(x, t) = 0$ when the velocity c is such that $c_*^+ < c < 0$. As far as we know, this kind of extinction waves was for the first time mentioned by K. Schumacher as *backward traveling fronts* in [17, p. 66: Example and Figure 3]. See also [4, 7, 25].

Finally, under weaker conditions on g, f , we get from Theorem 1.2 the following

Theorem 5.2. *Assume (\mathcal{F}) and let $u = \phi(x + ct)$ be a positive bounded solution of equation (10) satisfying $\liminf_{s \rightarrow -\infty} \phi(s) = 0$. Then $\phi(-\infty) = 0$, the critical speed c_*^+ is finite and $c \geq c_*^+$. A similar result holds when $\liminf_{s \rightarrow +\infty} \phi(s) = 0$. Hence, equation (10) does not have neither pulses nor semi-wavefronts propagating at the velocity $c \in (c_*^-, c_*^+)$.*

5.2. Nonlocal lattice equations [1, 8, 15, 16, 20, 23, 24].

Let consider semi-wavefronts $w_j(t) = u(j + ct)$ of the nonlocal lattice equation

$$w'_j(t) = D[w_{j+1}(t) - 2w_j(t) + w_{j-1}(t)] - dw_j(t) + \sum_{k \in \mathbb{Z}} \beta(j - k)g(w_k(t - r)), \quad j \in \mathbb{Z},$$

where $\beta(k) \geq 0$, $\sum_{k \in \mathbb{Z}} \beta(k) = 1$. Let $\pm\gamma_{\pm}^{\#} \geq 0$ be extended real numbers such that $\sum_{k \in \mathbb{Z}} \beta(k)e^{-zk}$ converges when $z \in \Gamma^{\#} := (\gamma_{-}^{\#}, \gamma_{+}^{\#})$ and is divergent when $\pm z > \pm\gamma_{\pm}^{\#}$. By Cauchy-Hadamard formula, $\gamma_{+}^{\#} = -\limsup_{k \rightarrow +\infty} k^{-1} \ln \beta(-k)$, where by convention $\ln(0) = -\infty$. A similar formula also holds for $\gamma_{-}^{\#}$. The wave profile u satisfies

$$cu'(x) = D[u(x + 1) + u(x - 1) - 2u(x)] - du(x) + \sum_{k \in \mathbb{Z}} \beta(k)g(u(x - k - cr)). \quad (11)$$

Let us take now $c \neq 0$. Then each positive bounded solution u of (11) satisfies (1) with $X = \{\tau_0, \tau_1\}$ and

$$K(s, \tau) = \begin{cases} D(H_{-1}(s) + H_1(s)), & \tau = \tau_0, \\ \sum_{k \in \mathbb{Z}} \beta(k)H_{k+cr}(s), & \tau = \tau_1, \end{cases} \quad g(s, \tau) = \begin{cases} s, & \tau = \tau_0, \\ g(s), & \tau = \tau_1, \end{cases}$$

$$H_{\tau}(t) = |c|^{-1}e^{-\frac{2D+d}{c}(t-\tau)}\chi_{\mathbb{R}_+}((\text{sign } c)(t-\tau)), \quad \chi(z, c) := \tilde{\chi}(z, c)(2D + d + cz)^{-1},$$

$$\tilde{\chi}(z, c) := d + 2D + cz - D(e^z + e^{-z}) - g'(0)e^{-crz} \sum_{k \in \mathbb{Z}} \beta(k)e^{-kz}, \quad d + 2D + cz > 0.$$

The following statement can be proved analogously to Lemma 6.1 in Appendix:

Lemma 5.3. *Assume that $\pm\gamma_{\pm}^{\#} > 0$ and that $g'(0) > d$. Then there exist real numbers $c_*^- < c_*^+$ such that, for every $c \in \mathfrak{C} := (-\infty, c_*^-] \cup [c_*^+, +\infty)$, equation $\chi(\lambda, c) = 0$ either (i) has exactly two real roots $\lambda_1(c) < \lambda_2(c)$ or (ii) has exactly one real root $\lambda_1(c)$. Furthermore, each $\lambda_j(c)$ is positive if $c \geq c_*^+$ and is negative if $c \leq c_*^-$. If $c \in (c_*^-, c_*^+)$, then $\chi(z, c) > 0$ for all $z \in (\gamma_{-}^{\#}, \gamma_{+}^{\#})$.*

Proof. See the proof of Lemma 6.1 below where it suffices to consider, instead of (12), the equation

$$d + 2D + cz - g'(0)e^{-crz} \sum_{k \in \mathbb{Z}} \beta(k)e^{-kz} = D(e^z + e^{-z}).$$

□

A formal computation shows that $\tilde{g}(s) = 2Ds/(2D + d)$, $\theta(s) = ds/(2D + d)$, $G(s) = g(s)/d$. Therefore, in complete analogy with the previous subsection, Theorem 1.5 yields the following

Theorem 5.4. *Let $G(s) = g(s)/d$ has properties 1-3 listed in Lemma 1.3 and $g(s) \leq g'(0)s$ for all $s \geq 0$. Then, for every $c \in \mathfrak{C} \setminus \{0\}$, the lattice equation has at least one semi-wavefront $u_j(t) = \phi_c(j + ct) \leq \zeta_2$. The profile ϕ_c shares every property mentioned in the conclusion part of Theorem 5.1 (with $f = \text{id}$).*

Theorem 5.4 extends [23, Theorem 3.1], [16, Theorem 2.1], [14, Theorem 5.4] and [9, Theorem 4.1] for non-monotone g and asymmetric β .

6. Appendix. Consider $\psi(z, c) = z^2 - cz - q + pe^{-zch} \int_{\mathbb{R}} K(s)e^{-zs}ds$, where $p > q$ and $K \geq 0$, $\int_{\mathbb{R}} K(s)ds = 1$.

Lemma 6.1. *Assume that $p > q > 0$ and that $\psi(z, c)$ is defined for all z from some maximal open interval $(a, b) \ni 0$. Then there exist real numbers $c_*^- < c_*^+$ such that, for every $c \in (-\infty, c_*^-] \cup [c_*^+, +\infty)$, equation $\psi(\lambda, c) = 0$ either (i) has exactly two real roots $\lambda_1(c) < \lambda_2(c)$ or (ii) has exactly one real root $\lambda_1(c)$. Furthermore, each $\lambda_j(c) \in (a, b)$ is positive if $c \geq c_*^+$ and is negative if $c \leq c_*^-$. If $c \in (c_*^-, c_*^+)$, then $\psi(z, c) > 0$ for all $z \in (a, b)$.*

Proof. Since $\psi_z''(z, c) > 0$, $z \in (a, b)$, we conclude that $\psi(z, c)$ is strictly convex with respect to z . Consequently, the equivalent equation

$$(H(z, c) :=)(q + cz - z^2)e^{zch} = p \int_{\mathbb{R}} e^{-zs} K(s)ds \quad (:= G(z)). \quad (12)$$

has at most two real roots. Since $\psi(0, c) = p - q > 0$, the convexity of ψ guarantees that these roots (whenever exist) are of the same sign. Next, we have that $G(0) = p > 0$, $G''(z) > 0$, $G(z) > 0$, $z \in (a, b)$. The left hand side of (12) increases to $+\infty$ [converges to 0] at each $z \in (0, b)$ when c tends to $+\infty$ [to $-\infty$ respectively] and the right hand side does not depend on c . Moreover, the left hand side of (12) increases with respect to c at every positive point z where $q + cz - z^2 > 0$. In consequence, if equation (12) has a positive root for some $c = c'$, then it does have a positive root for each $c > c'$. All this implies the existence of c_*^+ such that the equation (12) have either two positive roots $\lambda_1(c) \leq \lambda_2(c)$ or a unique positive root $\lambda_1(c)$ if and only if $c \geq c_*^+$. In fact, an easy analysis of (12) shows that the positive $\lambda_1(c)$ exists and depend continuously on c from the maximal open interval (c_*^+, ∞) .

Similarly, the left hand side of (12) increases to $+\infty$ [converges to 0] at each $z \in (a, 0)$ when c tends to $-\infty$ [to $+\infty$ respectively]. Moreover, the left hand side of (12) decreases with respect to c at every $z < 0$ where $q + cz - z^2 > 0$. This implies the existence of c_*^- such that the equation (12) has either two negative roots $\lambda_1(c) \leq \lambda_2(c)$ or a unique negative root $\lambda_2(c)$ if and only if $c \leq c_*^-$. Again the negative $\lambda_2(c)$ exists and depend continuously on $c \in (-\infty, c_*^-)$.

The above considerations also shows that c_*^- and c_*^+ are finite, and $c_*^- < c_*^+$. \square

Remark 4. With the unique exception ($c_*^- = -\infty$), all conclusions of Lemma 6.1 hold also true in the case when $(a, b) = (0, b)$, $b > 0$. To prove the finiteness of c_*^+ , it suffices to observe that for every positive δ there exists $c_1 < 0$ such that $H(z, c) < 0$ for all $z > \delta$, $c < c_1$ and $H(z, c) < p$ for all $z \in (0, \delta)$, $c < c_1$. A similar assertion (with $c_*^+ = +\infty$) is valid when $(a, b) = (a, 0)$, $a < 0$.

Acknowledgments. This research was supported by FONDECYT 1070127 (H. P.) and 1110309 (C. G. and S. T.). H. Prado and S. Trofimchuk acknowledge support from DICYT USACH 041133PC and CONICYT through PBCT program ACT-56, respectively. C. Gomez was also supported by CONICYT programs "Pasantías doctorales en el extranjero" and "Becas para Estudios de Doctorado en Chile".

REFERENCES

- [1] M. Aguerrea, C. Gomez and S. Trofimchuk, *On uniqueness of semi-wavefronts (Diekmann-Kaper theory of a nonlinear convolution equation re-visited)*, Math. Ann., DOI 10.1007/s00208-011-0722-8.
- [2] C. Atkinson and G. E. H. Reuter, *Deterministic epidemic waves*, Math. Proc. Cambridge Philos. Soc., **80** (1976), 315–330.
- [3] H. Berestycki, G. Nadin, B. Perthame and L. Ryzhik, *The non-local Fisher-KPP equation: travelling waves and steady states*, Nonlinearity, **22** (2009), 2813–2844 .
- [4] J. Coville, J. Dávila and S. Martínez, *Nonlocal anisotropic dispersal with monostable nonlinearity*, J. Diff. Eqns., **244** (2008), 3080–3118.
- [5] O. Diekmann, *Thresholds and travelling waves for the geographical spread of infection*, J. Math. Biol., **6** (1978), 109–130.
- [6] O. Diekmann and H. G. Kaper, *On the bounded solutions of a nonlinear convolution equation*, Nonlinear Anal., **2** (1978), 721–737.
- [7] J. Fang and X.-Q. Zhao., *Bistable traveling waves for monotone semiflows with applications*, preprint [arXiv:1102.4556v1](https://arxiv.org/abs/1102.4556v1).
- [8] J. Fang, J. Wei and X.-Q. Zhao, *Uniqueness of traveling waves for nonlocal lattice equations*, Proc. Amer. Math. Soc. **139** (2011), 1361–1373.
- [9] J. Fang, J. Wei and X.-Q. Zhao, *Spreading speeds and traveling waves for non-monotone time-delayed lattice equations*, Proc. R. Soc. A, **466** (2010) 1919–1934.
- [10] T. Faria, W. Huang and J. Wu, *Traveling waves for delayed reaction-diffusion equations with non-local response*, Proc. R. Soc. A , **462** (2006) 229–261.
- [11] K. Gopalsamy, N. Bantsur and S. Trofimchuk, *A note on global attractivity in models of hematopoiesis*, Ukrainian Math. J., **50** (1998) 5–12.
- [12] S.A. Gourley and J. W.-H. So, *Extinction and wavefront propagation in a reaction-diffusion model of a structured population with distributed maturation delay*, Proc. Royal Soc. of Edinburgh, **133A** (2003) 527–548.
- [13] S. A. Gourley, J. So and J. Wu, *Non-locality of reaction-diffusion equations induced by delay: biological modeling and nonlinear dynamics*, J. Math. Sciences, **124** (2004) 5119–5153.
- [14] X. Liang and X.-Q. Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications*, Comm. Pure Appl. Math., **60** (2007), 1–40.
- [15] S. Ma, *Traveling waves for non-local delayed diffusion equations via auxiliary equations*, J. Diff. Eqns., **237** (2007), 259–277.
- [16] S. Ma and X. Zou, *Existence, uniqueness and stability of travelling waves in a discrete reaction-diffusion monostable equation with delay*, J. Diff. Eqns., **217** (2005), 54–87.
- [17] K. Schumacher, *Travelling-front solutions for integro-differential equations. I* , J. Reine Angew. Math., **316** (1980), 54–70.
- [18] H.R. Thieme and X.-Q. Zhao, *Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models*, J. Diff. Eqns., **195** (2003) 430–470.
- [19] E. Trofimchuk, P. Alvarado and S. Trofimchuk, *On the geometry of wave solutions of a delayed reaction-diffusion equation*, J. Diff. Eqns., **246** (2009) 1422–1444.
- [20] E. Trofimchuk, M. Pinto and S. Trofimchuk, *Pushed traveling fronts in monostable equations with monotone delayed reaction*, preprint [arXiv:1111.5161v1](https://arxiv.org/abs/1111.5161v1).
- [21] D. Volkov and R. Lui, *Spreading speed and traveling wave solutions of a partially sedentary population*, IMA J. Appl. Math., **72** (2007) 801–816.
- [22] H. Weinberger, *Long time behavior of a class of biological models*, SIAM J. Math. Anal., **13** (1982) 353–396.
- [23] P. Weng, H. Huang and J. Wu, *Asymptotic speed of propagation of wave fronts in a lattice delay differential equation with global interaction*, IMA J. Appl. Math., **68** (2003) 409–439.
- [24] Z.-X. Yu, *Uniqueness of critical traveling waves for nonlocal lattice equations with delays*, Proc. Amer. Math. Soc., (2012), published on-line.
- [25] L. Zhang and B. Li, *Traveling wave solutions in an integro-differential competition model*, Discrete Contin. Dyn. Syst. - B, **17** (2012) 417–428.

E-mail address: cgomez@inst-mat.utalca.cl

E-mail address: humberto.prado@usach.cl

E-mail address: trofimch@inst-mat.utalca.cl